Spontaneous ordering against an external field in nonequilibrium systems

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We study the collective behavior of nonequilibrium systems subject to an external field with a dynamics characterized by the existence of non-interacting states. Aiming at exploring the generality of the results, we consider two types of models according to the nature of their state variables: (i) a vector model, where interactions are proportional to the overlap between the states, and (ii) a scalar model, where interaction depends on the distance between states. In both cases the system displays three phases: two ordered phases, one parallel to the field, and another orthogonal to the field; and a disordered phase. The phase space is numerically characterized for each model in a fully connected network. By placing the particles on a small-world network, we show that, while a regular lattice favors the alignment with the field, the presence of long-range interactions promotes the formation of the ordered phase orthogonal to the field.

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A rather general question considered in the framework of statistical physics of interacting particles (particles, spins, agents) is the competition between local particle-particle interactions (collective self-organization) and particle interaction with a global externally applied field or with a global mean field [1, 2]. Common wisdom answer to this question is that a strong external field dominates over local particle-particle interactions and orders the system by aligning particles with the broken symmetry imposed by the field. However, this is essentially an equilibrium concept which is not generally valid for generic non-potential interactions.

In the context of studies of collective phenomena in general systems of interacting particles, including biological and social systems, new forms of particle-particle and particle-field interactions are being considered. There are forms of interaction for which it turns out that a sufficiently intense external field induces disorder in the system [2, 3, 4], in contrast with the behavior in, for example, Ising-type systems. Other intriguing dynamical phenomenon is the collectively ordering in a state different from the one preferred by the forcing field. The external field might break symmetry in a given direction, but the system orders, breaking symmetry in a different direction. In this paper we examine this situation showing that these phenomena happen in two recently well studied non-equilibrium models [5, 6, 7, 8, 9]. What is common to these two models is that the particle-particle interaction rule is such that no interaction exists for some relative values of the states characterizing the particles that compose the system. A subsidiary question is the dependence of this phenomenon on the topology of the network of interactions. We show that the phenomenon is not found for particles interacting with its nearest neighbors in a regular lattice, but it occurs in a globally coupled system: it emerges as long range links in the network are introduced when going from the regular lattice to a random network via small world network [10].

The vector model, based in the dynamics of cultural dissemination of Axelrod model, consists of a set of N particles located at the nodes of an interaction network. The state of particle i is given by a F-component vector C_i^f $(f = 1, 2, \ldots, F)$ where each component can take any of q different values $C_i^f \in \{0, 1, \ldots, q-1\}$, leading to q^F equivalent states [5]. The external field, defined as a F-component vector $M^f \in \{0, 1, \ldots, q-1\}$, can interact with any of the particles in the system.

Starting from a random initial condition, at any given time, a randomly selected a particle can either interact with the external field or with one of its neighbors. The dynamics of the system is defined by iterating the following steps:

1. Select at random an particle i.
2. Select the *source of interaction*: with probability B the particle i interacts with the field, while with probability $(1-B)$ it interacts with one of its nearest neighbors j.
3. The overlap between the selected particle and the source of interaction is the number of shared components between their respective vector states, $d = \sum_{f=1}^{F} \delta_{C_i^f, X^f}$, where $X^f = M^f$ if the source of interaction is the field, or $X^f = C_j^f$ if i interacts with j. If $0 < d < F$, with probability d/F, choose h randomly such that $C_i^h \neq X^h$ and set $C_i^h = X^h$.

The strength of the field is represented by the parameter $B \in [0, 1]$ that measures the probability for the particle-field interactions. In the absence of an external field, $B = 0$, the system reaches a stationary configuration in any finite network, where for any pair of neighbors i and j, $d(i, j) = 0$ or $d(i, j) = F$. A *domain* is a set of connected particles with the same state. An homogeneous or ordered phase correspond to $d(i, j) = F$, $\forall i, j$, and obviously there are q^F equivalent configurations of this state. An inhomogeneous or disordered phase consist of the coexistence of several domains. In order to

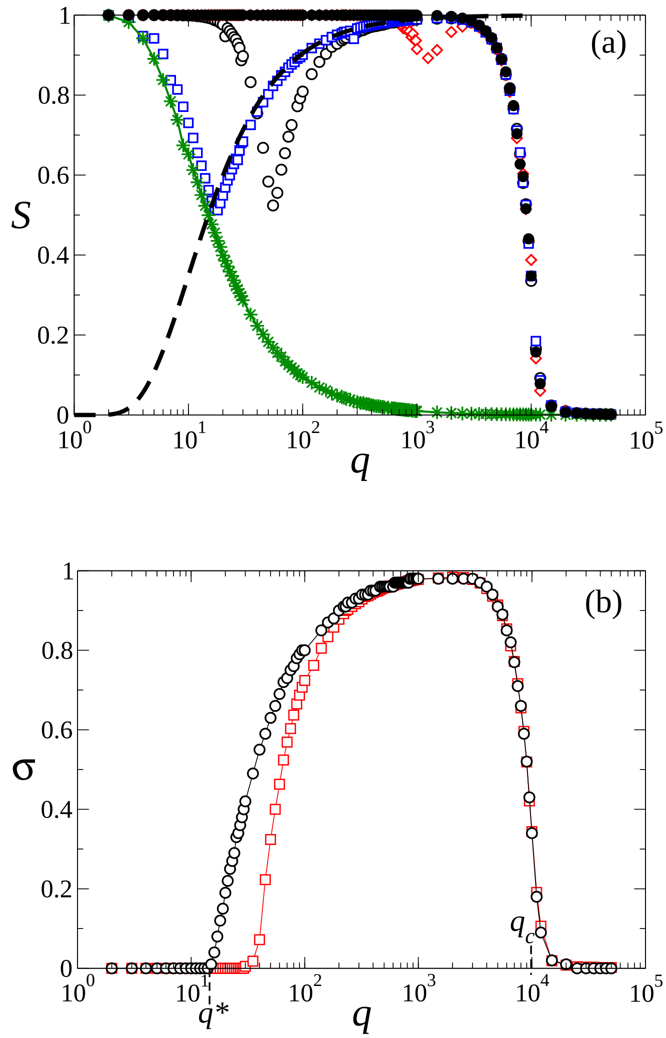

FIG. 1: (a) S as a function of q for the Axelrod model on a fully connected network for $B = 0$ (solid circles); $B = 0.005$ (diamonds); $B = 0.05$ (empty circles); $B = 0.5$ (squares); $B = 1$ (stars). The continuous line is the analytical curve $1-(1-1/q)^F$, while the dashed line corresponds to the curve $(1-1/q)^F$. (b) σ versus q for $B = 0.8$ (circles) and $B = 0.1$ (squares). The values of q_c and q^* are indicated for $B = 0.8$. Parameter values are $N = 2500$, $F = 10$.

characterize the ordering properties of this system, we consider as an order parameter the normalized average size of the largest domain S formed in the system. In finite networks the dynamics displays a critical point q_c that separates two phases: an ordered phase ($S \simeq 1$) for $q < q_c$, and a disordered phase ($S \ll 1$) for $q > q_c$ [6, 11, 12, 13].

First, we analyze the model in a fully connected network. In the absence of field, i.e. $B = 0$, the system spontaneously reaches a homogeneous state for values $q < q_c \approx 10^4$ (Fig. 1-a). For $B \to 0$ and $q < q_c$, the field M^f is able to impose this homogeneous state to the system, as in a two-dimensional network [4]. For $B = 1$, the particles only interact with the external field; in this case only those particles that initially share at least one component of their vector states with the components of M^f will converge to the field state M^f. The fraction of particles that do not share any component with M^f is given by $(1-1/q)^F$; thus the fraction of those particles that converge to M^f is $1-(1-1/q)^F$. Figure 1 a shows both the numerically calculated values of S as well as the analytical curve of S_M versus q, for fixed $B = 1$. Both quantities agree very well, indicating that the largest domain in the system possesses a vector state equal to that of the external field when $B = 1$.

For intermediate values of B, the spontaneous order emerging in the system for parameter values $q < q_c$ due to the particle-particle interactions competes with the order being imposed by the field. This competition is manifested in the behavior of the order parameter S which displays a sharp local minimum at a value $q^*(B) < q_c$ that depends on B, while the value of q_c is found to be independent of the intensity B, as shown in Fig. 1 a. To understand the nature of this minimum, we plot in Fig. 1 b the quantity $\sigma = S - S_M$, as a function of q, where S_M is the normalized average size of the largest domain displaying the state of the field M^f. For $q < q^*(B)$ the largest domain corresponds to the state of the external field, $S = S_M$, and thus $\sigma = 0$. For $q > q^*(B)$, the largest domain no longer corresponds to the state of the external field M^f but to other state non-interacting with the external field, i.e., $S > S_M$, and $\sigma > 0$. The value of $q^*(B)$ can be estimated for the limiting case $B \to 1$, for which $S_M \approx 1-(1-1/q)^F$ and the largest domain different from the field is $S \approx 1 - S_M$. Therefore the condition $S = S_M$ yields $q^*(B \to 1) = \left[1-(1/2)^{1/F}\right]^{-1}$. For $F = 10$ it gives $q^*(B \to 1) = 15$ in good agreement with the numerical results. The order parameter σ reaches a maximum at some value of q between q^* and q_c above which order decreases in the system and both $S \to 0$, $S_M \to 0$. As a consequence, σ starts to decrease.

The collective behavior of the vector model on a fully connected network subject to an external field can be characterized by three phases on the space of parameters (q, B), as shown in Fig. 2: (I) an ordered phase induced by the field for $q < q^*$, for which $\sigma = 0$ and $S = S_M \neq 0$; (II) an ordered phase in a state different from that of the field for $q* < q < q_c$, for which σ increases and $S > S_M$; and (III) a disordered phase for $q > q_c$, for which σ decreases and $S \to 0$, $S_M \to 0$.

For parameter values $q < q_c$ for which the system orders due to the interactions among the particles, a sufficiently weak external field is able to impose its state to the entire system (phase I). However, if the probability of interaction with the field B exceeds a threshold value, the system spontaneously orders in a state different from that of the field (phase II).

Continuous states based on bounded interactions provide other instances of a nonequilibrium systems where induced and spontaneous order compete in the presence of an external field. Consider, for example, the bounded

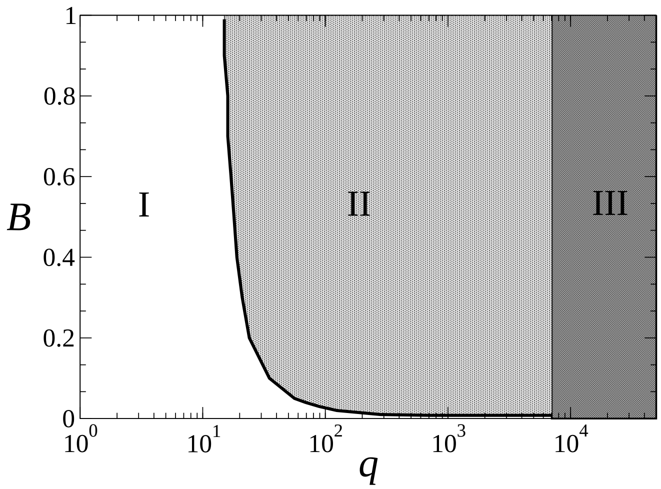

FIG. 2: Phase space on the plane (q, B) for the vector model on a fully connected network subject to an external field, with fixed $F = 10$. Regions where the phases I, II, and III occur are indicated.

confidence model [7]. It consists of a population of N particles where the state of particle i is given by a real number $C_i \in [0, 1]$. We introduce an external field $M \in [0, 1]$ that can interact with any of the particles in the system. The strength of the field is again described by a parameter $B \in [0, 1]$ that measures the probability for the particle-field interactions, as in the vector model.

We start from a uniform, random initial distribution of the states of the particles. At each time step, an particle i is randomly chosen;

1. with probability B, particle i interacts with the field M: if $|C_i - M| < d$, then

$$C_i^{t+1} = \frac{1}{2}(M + C_i^t) \; ; \tag{1}$$

2. otherwise, a nearest neighbor j is selected at random: if $|C_i - C_j| < d$ then:

$$C_i^{t+1} = C_j^{t+1} = \frac{1}{2}(C_j^t + C_i^t) \; . \tag{2}$$

The parameter d defines a threshold distance for interaction and the remainder we fix $M = 1$.

We calculate the normalized average size of the largest domain S in the system as a function of $1-d$, for different values of B, as shown in Fig. 3(a). For $B = 0$, the system spontaneously reaches a homogeneous state $C_i = 0.5$, $\forall i$, characterized by $S = 1$, for values $1 - d < 1 - d_c \approx 0.77$, with $d_c \approx 0.23$ [7]; while for $1-d > 1-d_c$ several domains are formed yielding $S < 1$.

For $B = 1$ particles only interact with the field; in this case the value of M is imposed on the largest domain whose normalized size increases with the threshold, i.e. $S = d$. For intermediate values of B, the spontaneous order emerging in the system for values of $1 - d < 1 - d_c$

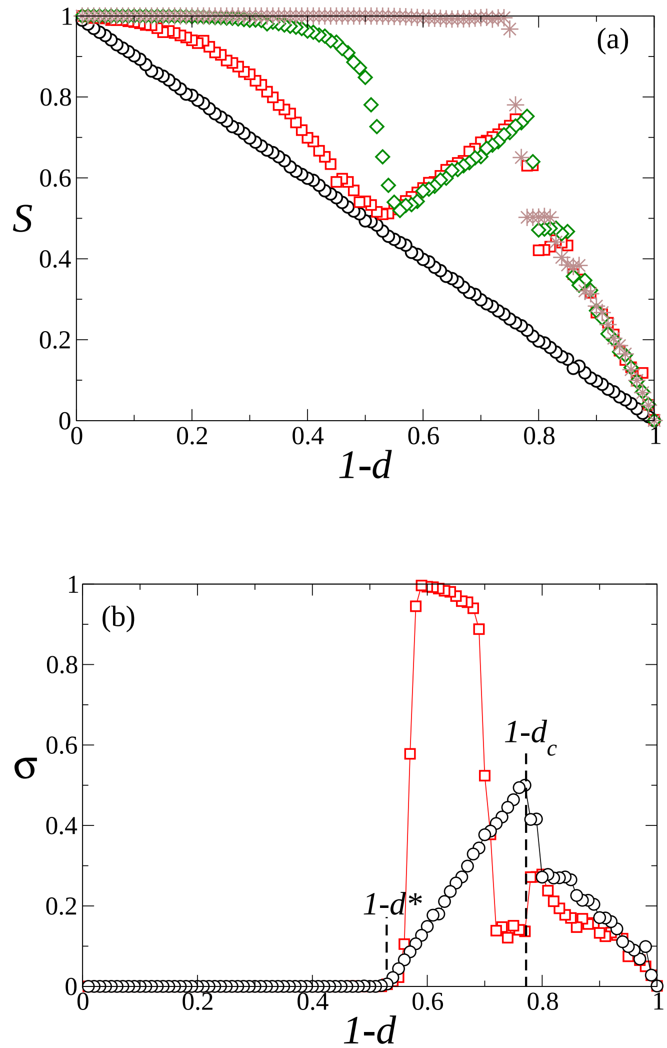

FIG. 3: (a) S versus $1 - d$ for the continuous model for $B = 0$ (stars); $B = 0.5$ (diamonds); $B = 0.8$ (squares); $B = 1$ (circles). (b) σ vs. d for $B = 0.8$ (circles) and $B = 0.1$ (squares). The values of $1 - d_c$ and $1 - d^*$ are indicated for $B = 0.8$. Size of the system is $N = 2500$.

due to the interactions between the particles competes with the order being induced by the field. The quantity S exhibits a sharp local minimum at a value $1 - d = 1 - d^* < 1 - d_c$, as shown in Fig. 3(a). In Fig. 3(b) we plot the order parameter $\sigma = S - S_M$ as a function of $1 - d$, for different values of B. For $1 - d < 1 - d^*$ the largest domain reaches a state equal to M, that is $S = S_M$, and thus $\sigma = 0$. At $1 - d = 1 - d^*$, the state of the field no longer corresponds to the largest domain, i.e., $S > S_M$, and σ starts to increase as $1-d$ increases. For a small value of B, the quantity σ reaches a maximum close to one, indicating that the spontaneously formed largest domain almost occupies the entire system, i.e., the field is too weak to compete with the attracting homogeneous state $C_i = 0.5$, $\forall i$. However, when B is increased, the maximum of σ is about 0.5, i.e., the attraction of the

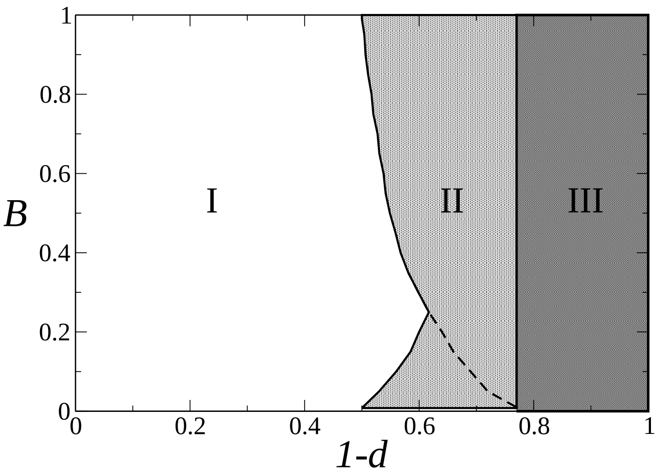

FIG. 4: Phase space on the plane $(1-d, B)$ for the scalar model on a fully connected network subject to an external field. Regions where the phases I, II, and III occur are indicated. The dashed line in phase II separates region where the maximum of $\sigma \to 1$ (below this line) from the region where $\sigma \leq 0.5$ (above this line).

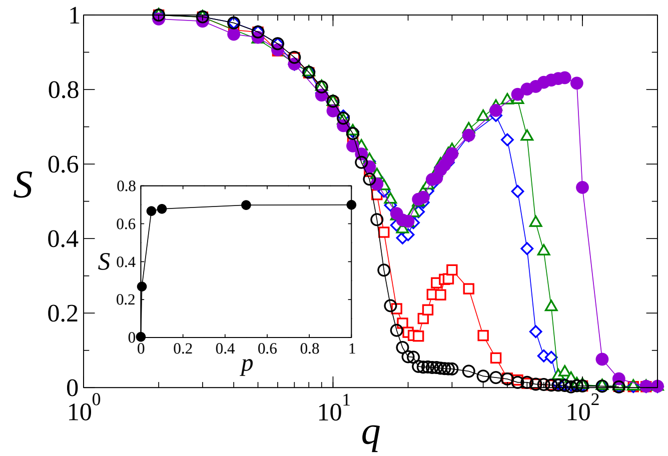

FIG. 5: S versus q in the vector model on a small world network with $\langle k \rangle = 4$, $B = 0.5$, $F = 3$, for different values of the probability p: $p = 0$ (empty circles), $p = 0.005$ (squares), $p = 0.05$ (diamonds), $p = 0.1$ (triangles), $p = 1$ (solid circles). Inset: S vs. p for fixed values $q = 40 > q^*$ and $B = 0.5$.

field $M = 1$ increases and the size of the domain with a state equal to M is not negligible in relation to the size of the largest domain. In contrast, in the vector model the maximum $\sigma \to 1$ in the region $q* < q < q_c$, independently of the value of B.

The value of d^* in the scalar model depends on B and it can be estimated for $B \to 1$. In this case, $S_M \approx d$ and $S \approx 1 - d$; thus the condition $S = S_M$ yields $d^* \approx 0.5$ when $B \to 1$. The quantity σ reaches a maximum at the value $1-d \approx 1-d_c$, above which disorder increases in the system, and both S and S_M decrease. As a consequence, σ decreases for $1 - d > 1 - d_c$.

As in the vector model, the collective behavior exhibited by the scalar model on a fully connected network subject to an external field can be characterized by three phases: (I) an ordered phase parallel to the field for $1 - d < 1 - d^*$, for which $\sigma = 0$ and $S = S_M \neq 0$; (II) a ordered phase for $1 - q^* < 1 - d < 1 - d_c$, for which σ increases and $S > S_M$; and (III) a disordered phase for $1 - d > 1 - d_c$, for which σ decreases and both S and S_M decrease. Figure 4 shows the phase diagram on the plane $(1-d, B)$ for the scalar model subject to an external field. The continuous curve separating phases I and II gives the dependence $d^*(B)$.

Short range interactions. To analyze the role of the connectivity on the emergence of an ordered phase orthogonal to the external field, we consider a small-world network [10], where the rewiring probability can be varied in order to introduce long-range interactions between the particles. We start from a two-dimensional lattice sites with nearest-neighbor interactions. Each connection is rewired at random with probability p. The value $p = 0$ corresponds to a regular network, while $p = 1$ corresponds to a random network with $\langle k \rangle = 4$.

Figure 5 shows the order parameter S as a function of q in the vector model defined on this network for different values of the rewiring probability p and for a fixed value of the intensity of the field B. The critical value q_c where the order-disorder transition takes place increases with p, which is compatible with the large value of q_c observed in a fully connected network. When the long-range interactions between particles are not present, i.e. $p = 0$, the external field is able to impose its state to the entire system for $q < q_c$. Spontaneous ordering different from the state of the external field appears as the probability of having long-range interactions increases. The size of this alternative largest domain increases with p, but it does not grow enough to cover the entire system (see inset in Fig. 5). Increasing the rewiring probability in the scalar model also produces a behavior similar to the vector model. Thus, in systems whose dynamics is based on a bound for interaction, the presence of long-range connections facilitates the emergence of spontaneous ordering not associated to the state of an applied external field.

In summary, we have studied the collective behavior of nonequilibrium systems with non-interacting states and subject to an external field. We have considered two models on a fully connected network that share a common feature: the existence of non-interacting states. In both cases we have found three phases depending on parameter values: two ordered phases, one having a state equal to the external field, an another ordered phase, consisting of a large domain with a state orthogonal to the field; and a disordered phase. The occurrence of an ordered phase with a state orthogonal to the field is enhanced by the presence of long range connections in the underlying network. We have verified that this alternative ordered phase also appears when the models consid-

ered here are defined on a scale-free network.

The emergence of an ordered phase with a state different from that of an external field may be relevant in social systems as well as in many biological systems having motile elements, such as swarms, fish schools, and bird flocks [14], whose dynamics usually possess a bound condition for interaction. Thus one may expect that this phenomenon should arise in large class of nonequilibrium systems in the presence of an external source for interaction.

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